Math2050A Term1 2017
Tutorial 7, Nov 2

The Thomae's function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } |x| = \frac{m}{n} \text{ with } \gcd(m,n) = 1. \end{cases}$$

is a function continuous precisely on $\mathbb{R} \setminus \mathbb{Q}$.

Now, we may ask if there is $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous precisely on $\mathbb{Q}$. The answer is "No". You are recommended to consult the lecture notes titled "No function cts at rational only" in webpage of math2050b (2016/17) by Professor Ng. I try to copy it here.You may skip it.

Suppose such f exists. Write $\{r_1, r_2, ...\} = \mathbb{Q}$.
Choose an open interval $I_1 := (r_1 - \epsilon, r_1 + \epsilon)$ such that $|f(x) - f(r_1)| < 1$ for all $x \in I_1$. We may assume the end points of I_1 to be irrational. Let r_{n_2} to be the next term inside the interval I_1, which means that $r_k \notin I_1$ for any $1 < k < n_2$, but $r_{n_2} \in I_1$. Then, we choose an open interval $I_2 := (r_{n_2} - \epsilon_2, r_{n_2} + \epsilon_2)$ such that

1. $I_2 \subset I_1$
2. $r_1 \notin I_2$
3. $|f(x) - f(r_{n_2})| < \frac{1}{2}$ for all $x \in I_2$.
4. the end points of I_2 are irrational

Notice that $r_i \notin [r_{n_2} - \epsilon_2, r_{n_2} + \epsilon_2]$ for any $i < n_2$.
Given $I_k = (r_{n_k} - \epsilon_k, r_{n_k} + \epsilon_k)$, let $r_{n_{k+1}}$ to be the next term inside I_k. Then, we choose an open interval $I_{k+1} := (r_{n_{k+1}} - \epsilon_{k+1}, r_{n_{k+1}} + \epsilon_{k+1})$ such that

1. $I_{k+1} \subset I_k$
2. $r_{n_k} \notin I_{k+1}$

3. $|f(x) - f(r_{n_{k+1}})| < \frac{1}{k+1}$ for all $x \in I_{k+1}$.

4. the end points of I_{k+1} are irrational

Then, we have $r_i \notin [r_{n_{k+1}} - \epsilon_{k+1}, r_{n_{k+1}} + \epsilon_{k+1}]$ for any $i < n_{k+1}$.
The process doesn't terminate.

Applying nested intervals theorem on $J_k := [r_{n_k} - \epsilon_k, r_{n_k} + \epsilon_k]$, then $\cap_{k=1}^{\infty} J_k \neq \emptyset$. Since every rational number r_i is not inside J_{i+1}, every member in $\cap_{k=1}^{\infty} J_k$ is irrational, say $\xi \in \cap_{k=1}^{\infty} J_k$. The above construction seems not guarantee the continuity of f at ξ because ξ may be end point of each J_k. However, it can be resolved when we replace those requirements in construction : $I_2 \subset I_1$ by $J_2 \subset I_1$ and $I_{k+1} \subset I_k$ by $J_{k+1} \subset I_k$ for each k, while others remain unchanged. Restart the construction and pick $\xi' \in \cap_{k=1}^{\infty} J_k$.

Now, f is continuous at ξ' because $\xi' \in \cap_{k=2}^{\infty} J_k \subset \cap_{k=1}^{\infty} I_k$. Let $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. $\xi' \in I_{2N}$ implies that there is $\delta > 0$ such that $(\xi' - \delta, \xi' + \delta) \subset I_{2N}$. If $|w - \xi'| < \delta$, then $w \in I_{2N}$, hence $|f(w) - f(r_{n_{2N}})| < \frac{1}{2N} < \frac{\epsilon}{2}$. Together with $\xi' \in I_{2N}$, and by triangle inequality, $|f(\xi') - f(w)| < \epsilon$.

Question:
Determine if the following functions are uniformly continuous:

(a) $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$,

(b) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$,

(c) $f : [0, M) \to \mathbb{R}$ defined by $f(x) = x^2$, where $M > 0$,

(d) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$,

(e) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x^2 + 1}$,

(f) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x^2)$.